§ 4.5* Maximum Likelihood (ML) Decoding

Maximum Likelihood (ML) Decoding: Given a received word / symbol sequence $\bar{y}$, the codeword $\bar{c}$ (or message $\bar{u}$) that maximizes the channel transition probability $P(\bar{y}|\bar{c})$ is the decoding output which is denoted as $\hat{\bar{c}}$ (or $\hat{\bar{u}}$). That says

$$\hat{\bar{c}} = \underset{\bar{c} \in \mathcal{C}}{\operatorname{argmax}} P(\bar{y}|\bar{c}).$$

- Based on Bayes' theorem, the a posteriori probability can be determined as

$$P(\bar{c}|\bar{y}) = \frac{P(\bar{y}|\bar{c})P(\bar{c})}{P(\bar{y})}.$$

Maximum A Posteriori (MAP) Decoding: Given $\bar{y}$, the codeword $\bar{c}$ (or message $\bar{u}$) that maximizes the MAP $P(\bar{c}|\bar{y})$ is the decoding output. That says

$$\hat{\bar{c}} = \underset{\bar{c} \in \mathcal{C}}{\operatorname{argmax}} P(\bar{c}|\bar{y}).$$

- By assuming equiprobable codeword as $P(\bar{c}) = |\mathcal{C}|^{-1}$, the ML decoding output coincides with the MAP decoding.

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Union Bound

- Union bound can be used to characterize the ML decoding performance of codes, which requires knowledge of the code's weight spectrum (distribution of codewords of different weights).

- The codeword $c_1^N = \bar{c} = \{c_1, c_2, \cdots, c_N\} \in \mathcal{C}$ of a linear block code has discrete weight values, denoted as $\{d_0, d_1, d_2, \dots, d_s\}$, where $d_0 = 0$, $d \leq d_i \leq N$ and $i = 1, 2, \dots, s$. The number of codewords with weight d_i is denoted as A_{d_i}. Hence, weight spectrum is $\{\mathrm{A}_{d_i}, \forall i\}$.

- Union upper bound on a linear block code's ML decoding frame error rate (FER) over the AWGN channel is

$$P_{\mathrm{ML,e}} \leq \sum_{d_i=d}^{d_s} \mathrm{A}_{d_i} Q\left(\frac{\sqrt{d_i}}{\sigma}\right) = \sum_{d_i=d}^{d_s} \mathrm{A}_{d_i} Q\left(\frac{\sqrt{r d_i E_b}}{\sigma}\right).$$

- code rate (r)
- energy of each info. bit (E_b)
- noise standard deviation (σ)

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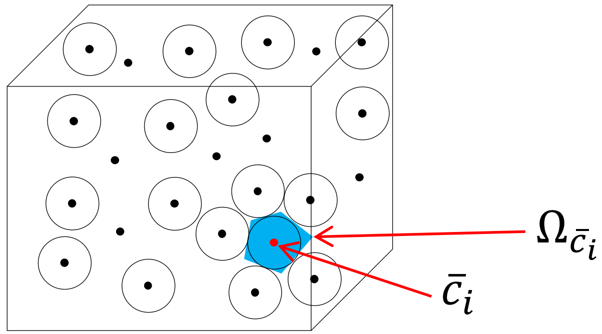
- Proof:

The function $\mathbb{1}(\cdot)$ denotes the indicator function, where $\mathbb{1}(\text{true}) = 1$ and $\mathbb{1}(\text{false}) = 0$. Then,

$$P_{\text{ML,e}} = \sum_{\bar{c}\in\mathcal{C}} \sum_{\bar{y}\in\mathcal{Y}} \Pr(\bar{c},\bar{y}) \cdot \mathbb{1}(\text{Decoder}_{\text{ML}}(\bar{y}) \neq \bar{c}).$$

The set $\Omega_{\bar{c}_i}$ is defined as $\Omega_{\bar{c}_i} = \left\{\bar{y} \middle| \Pr(\bar{y}|\bar{c}_i) > \Pr\left(\bar{y}\middle|\bar{c}_{i'}\right), \forall i \neq i'\right\}$, which represents the space of all received signals $\bar{y}$ that will be decoded as the codeword $\bar{c}_i$ under the ML decision rule. The set $\mathcal{Y}$ is n-dimensional real vector space, and $\mathcal{Y} = \cup_{\bar{c}_i\in\mathcal{C}} \Omega_{\bar{c}_i}$.



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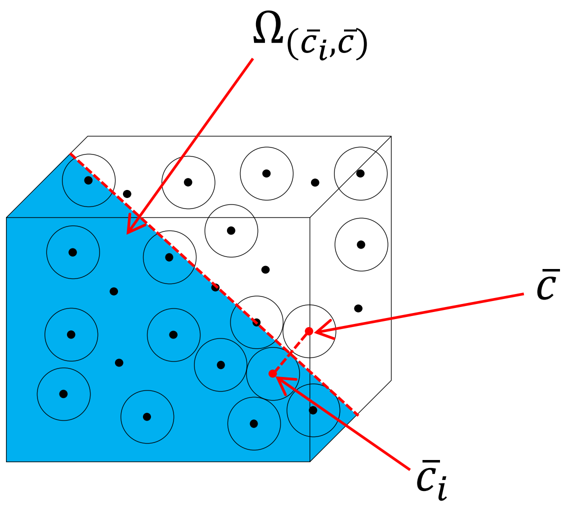
By symmetry, let $\bar{c}$ be $\bar{0}$. We have

$$P_{\text{ML,e}} = \sum_{\bar{c}\in\mathcal{C}} \sum_{\bar{y}\in\mathcal{Y}\setminus\Omega_{\bar{c}}} \Pr(\bar{c},\bar{y}) = \Pr(\bar{c}) \sum_{\bar{c}\in\mathcal{C}} \sum_{\bar{y}\in\cup_{\bar{c}_i\neq\bar{c}}\Omega_{\bar{c}_i}} \Pr(\bar{y}|\bar{c}) = \sum_{\bar{y}\in\cup_{\bar{c}_i\neq\bar{c}}\Omega_{\bar{c}_i}} \Pr(\bar{y}|\bar{c}).$$

The set $\Omega_{(\bar{c}_i,\bar{c})}$ is further defined as $\Omega_{(\bar{c}_i,\bar{c})} = \{\bar{y}|\Pr(\bar{y}|\bar{c}_i) > \Pr(\bar{y}|\bar{c}),\ \bar{c}_i \neq \bar{c}\} \supset \Omega_{\bar{c}_i}$, which represents the space of $\bar{y}$ that will be decoded as $\bar{c}_i$ instead of $\bar{c}$. Hence

$$\begin{aligned} P_{\text{ML,e}} &= \sum_{\bar{y}\in\cup_{\bar{c}_i\neq\bar{c}}\Omega_{\bar{c}_i}} \Pr(\bar{y}|\bar{c}) \\ &= \sum_{\bar{y}\in\cup_{\bar{c}_i\neq\bar{c}}\Omega_{(\bar{c}_i,\bar{c})}} \Pr(\bar{y}|\bar{c}) \\ &\leq \sum_{i=1}^{2^K-1} \sum_{\bar{y}\in\Omega_{(\bar{c}_i,\bar{c})}} \Pr(\bar{y}|\bar{c}). \end{aligned}$$



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Under AWGN channel and BPSK modulation,

$$\sum_{\bar{y}\in\Omega_{(\bar{c}_i,\bar{c})}} \Pr(\bar{y}|\bar{c}) = Q\left(\sqrt{\frac{2E_c d_{\text{Ham}}(\bar{c}_i,\bar{c})}{N_0}}\right) = Q\left(\frac{\sqrt{d_{\text{Ham}}(\bar{c}_i,\bar{c})}}{\sigma}\right) = Q\left(\frac{\sqrt{d_{\text{Ham}}(\bar{c}_i,\bar{0})}}{\sigma}\right).$$

pairwise error probability ↖   assuming $E_c = 1$ ↖

Then,

$$\sum_{i=1}^{2^K-1} \sum_{\bar{y}\in\Omega_{(\bar{c}_i,\bar{c})}} \Pr(\bar{y}|\bar{c}) = \sum_{\bar{y}\in\Omega_{(\bar{c}_1,\bar{c})}} \Pr(\bar{y}|\bar{c}) + \sum_{\bar{y}\in\Omega_{(\bar{c}_2,\bar{c})}} \Pr(\bar{y}|\bar{c}) + \cdots + \sum_{\bar{y}\in\Omega_{(\bar{c}_{2^K-1},\bar{c})}} \Pr(\bar{y}|\bar{c}).$$

$$E_c = 1 = rE_b \quad \Rightarrow \quad P_{\text{ML,e}} \le \sum_{d_i=d}^{d_s} \mathrm{A}_{d_i} Q\left(\frac{\sqrt{r d_i E_b}}{\sigma}\right).$$

□